2020 B

March 30

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A parametric surface is a map

$$\vec{r} : [a,b] \times [c,d] \longrightarrow \mathbb{R}^3$$

$$\vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v)) \quad \text{or}$$
$$= x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k} \quad \text{or}$$
$$= f(u,v)\,\hat{i} + g(u,v)\,\hat{j} + h(u,v)\,\hat{k}$$

$\vec{r}$ is C^1 (or smooth) if the fcns f, g, h are C^1 (or smooth)

It is <u>regular</u> if

$$\vec{r}_u \times \vec{r}_v \neq \vec{0}, \quad \forall (u,v) \in (a,b) \times (c,d).$$

It means that

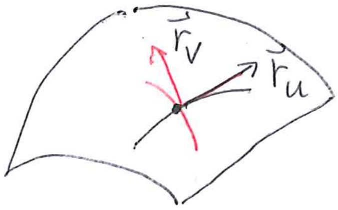
① $\vec{r}_u, \vec{r}_v \neq \vec{0}$, ie, fix v, the p. curve $u \mapsto \vec{r}(u,v)$ is regular,

fix u, $v \mapsto \vec{r}(u,v)$ is regular.

and

② $\vec{r}_u$ and $\vec{r}_v$ are linearly indept.

<u>Note</u> $[a,b]\times[c,d]$ can be replaced by region $D \subset \mathbb{R}^2$



e.g. Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.

Spherical coordinates suggests to use (φ, θ) as (u,v)

$$(\varphi, \theta) \longmapsto a \sin\varphi \cos\theta\,\hat{i} + a \sin\varphi \sin\theta\,\hat{j} + a\cos\varphi\,\hat{k}$$

$[0,\pi] \times [0, 2\pi]$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\varphi\cos\theta\,\hat{i} + a^2\sin^2\varphi\sin\theta\,\hat{j} + a^2\cos\varphi\sin\varphi\,\hat{k}$$

$$|\vec{r}_\varphi \times \vec{r}_\theta|^2 = a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\cos^2\varphi\sin^2\varphi$$

$$= a^4\sin^4\varphi + a^4\cos^2\varphi\sin^2\varphi$$

$$= a^4\sin^2\varphi$$

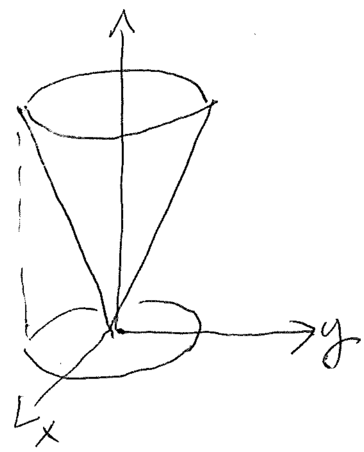
$\therefore\ |\vec{r}_\varphi \times \vec{r}_\theta| = a^2|\sin\varphi| = a^2\sin\varphi \geq 0$

> 0 on $(0,\pi)\times(0,2\pi)$

$\therefore$ this parametrization is a regular one.

e.g. Find a parametrization of the circular cone

$$z = \sqrt{x^2+y^2}, \quad 0 \leq z \leq 1.$$



This is the graph of the fcn

$$z = \sqrt{x^2+y^2}$$

on the unit disk D_1. Use

$$(x,y) \longmapsto (x, y, \sqrt{x^2+y^2})$$
$$D_1$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\,\hat{i} - \frac{x}{\sqrt{x^2+y^2}}\,\hat{j} + \hat{k}$$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{2} > 0 \ \therefore$ regular.

In general, whenever the surface is a graph of some fcn f over D, a regular parametrization is

$$(x, y) \longmapsto x\hat{i} + y\hat{j} + f(x,y)\hat{k}.$$

The <u>surface area</u> of a surface is defined to be

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, dA(u,v)$$

when $\vec{r}$ is a regular parametrization of the surface.

e.g. Find the surface area of the surface $x^2+y^2+z^2=a^2$.

From the e.g. above,

$$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2 \sin\varphi$$

$$\therefore \text{ surface area } = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\varphi \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi} a^2 \sin\varphi \, d\varphi$$

$$= 4\pi a^2.$$

Another approach. The upper hemisphere is

$$(x, y) \longmapsto (x, y, \sqrt{a^2-x^2-y^2})$$
$$D_a$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -x/\sqrt{\ } \\ 0 & 1 & -y/\sqrt{\ } \end{vmatrix}$$

$$= \frac{x}{\sqrt{\ }}\hat{i} + \frac{y}{\sqrt{\ }}\hat{j} + \hat{k}.$$

$$|\vec{r}_x \times \vec{r}_y| = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

$\therefore$ surface area of sphere

$$= 2 \iint_{D_a} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA(x,y)$$

Changing to polar coor.

$$= 2 \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= 4\pi a \int_0^{a^2} \frac{dt}{\sqrt{a^2-t}} \qquad (t=r^2)$$

$$= 4\pi a^2 \ \#$$

e.g. Find the surface area of the cone $z=\sqrt{x^2+y^2}$, $0 \le z \le 1$.

As above, $|\vec{r}_x \times \vec{r}_y| = \sqrt{2}$.

$$\text{surface area} = \iint_{D_1} \sqrt{2}\, dA(x,y)$$

$$= \sqrt{2} \times |D_1| \quad (\,|D_1| \text{ is the area of the unit disk})$$

$$= \sqrt{2}\,\pi. \ \#$$